Correlation length exponent in the three-dimensional fuse network

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We present numerical measurements of the critical correlation length exponent ν in the three-dimensional fuse model. Using sufficiently broad threshold distributions to ensure that the system is the strong-disorder regime, we determine ν to be $\nu=0.83\pm0.04$ based on analyzing the fluctuations of the survival probability. This value is different from that of ordinary percolation, which is 0.88.

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It is already twenty years since the publication of the first experimental evidence of scaling in the morphology of brittle fractures [1]. About seven years later it was proposed that not only is there scaling, but the scaling properties are *universal*, in the sense that they do not depend on material properties [2,3]. There is now mounting evidence for this hypothesis, which may be expressed as the scaling invariance

$$\pi(z;x,y) = \lambda^{\zeta}\pi(\lambda^{\zeta}z;\lambda x,\lambda y), \tag{1}$$

of π, which is the probability density that at position (x,y) in the average fracture plane, the fracture is at height z given that it is at $z=0$ at (0, 0), with ζ as the universal roughness exponent having a value very close to 0.80 for a large class of materials. One experimentally important consequence of this scaling is that the average fracture width w scales as

$$w \sim L^{\zeta}, \tag{2}$$

where L is the linear size of the average fracture plane.

Ever since the proposal of universality, it has remained a theoretical challenge to explain this value. Recently, it was suggested by Hansen and Schmittbuhl that it has its origin in the fracture process being a correlated percolation process [4]. The essence of the argument is based on the existence of a localization length l and a correlation length ξ that grows during the breakdown process. The localization length depends on the disorder in the material: Stronger disorder means larger localization length. Whether the localization length diverges for large but finite disorder or it only reaches this limit for infinite disorder is still a question that has not been answered definitively. Toussaint and Hansen [5] find within a mean field approximation for the two-dimensional fuse model that an infinite localization length is only obtained in the limit of infinite disorder limit in an infinitely large system. In the present paper, we claim that infinite localization length indeed is attained for finite disorder in the three-dimensional fuse model.

For correlation lengths ξ much smaller than the localization length l, Hansen and Schmittbuhl [4] assumed a relation

$$\xi \sim |p - p_c|^{-\nu}, \tag{3}$$

where p is the local damage density and p_c is the damage density at failure. This relation is taken directly from percolation theory. The reason it is only valid for large localization lengths l is that p is assumed to be spatially stationary (meaning that the statistical distribution of p values is independent of position). The correlation length exponent ν has the value $4/3$ in two-dimensional percolation and 0.88 in three-dimensional percolation [7]. It is by no means given that ν should be the same in the brittle fracture problem—and Toussaint and Pride suggest that it is equal to 2 [8]. It was suggested by Hansen and Schmittbuhl based on numerical simulations that the two-dimensional fuse model has $\nu=4/3$, i.e., the value found in the pure percolation problem. Large-scale simulations by Nukala *et al.* [6], however, show that the value is close to $\nu=1.56$. When the correlation length approaches the localization length l, gradients develop in the damage—p can no longer be regarded as spatially stationary—and using arguments from gradient percolation [9], Hansen and Schmittbuhl suggested the relation

$$\zeta = \frac{2\nu}{1+2\nu}. \tag{4}$$

Recent numerical calculations gives $\zeta=0.74\pm0.03$ [10], while the value inferred from Nukala *et al.* [6] suggests $\zeta=0.76$ when using Eq. (4).

It is the aim of this paper to measure ν in the three-dimensional fuse model. We find the value $\nu=0.83\pm0.04$. The three-dimensional percolation value $\nu=0.88$ is slightly outside the error bars of this value. However, we believe it is unlikely that the three-dimensional fuse model is in the universality class of ordinary percolation as long as the two-dimensional one clearly is not. The roughness exponent ζ was measured by Batrouni and Hansen [12] to be $\zeta=0.62\pm0.05$. Using Eq. (4) with $\nu=0.83$, we find $\zeta=0.62$. Hence the value for ν we report here is consistent with the roughness exponent measured in Ref. [12]. We note, however, that $\zeta=0.62$ is far from the value reported by Räisänen *et al.* [13], claiming it to be identical to the minimal energy result, $\zeta=0.41\pm0.02$ [14].

The fuse model that we study consists of an oriented simple cubic lattice. As in Ref. [12], we use *periodic boundary conditions in all directions* [15] and the average current flows in the (1,1,1) direction. Each bond is an Ohmic resistor up to a threshold value. When this value is reached, the resistor turns irreversibly into an insulator. The threshold values are drawn from a spatially uncorrelated probability density $p(t)$. A voltage drop equal to unity is set up across the lattice along a given plane orthogonal to the (1,1,1) direction.

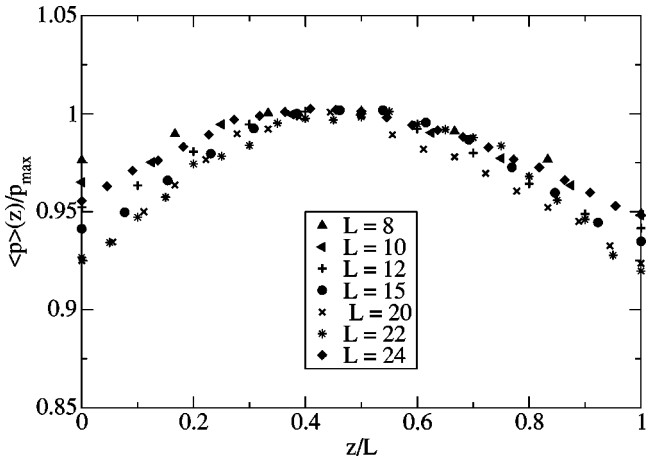

FIG. 1. Normalized histogram where each bin is averaged over a plane orthogonal to the (1,1,1) direction for $D=10$. The position of the damage zone for each sample is adjusted so that its *center of mass* is at the center of the plot before averaging. This is in contrast to the damage profile averaging performed in Ref. [18] where the *maximum* of each damage profile was used as center point. The latter procedure will always produce a cusp.

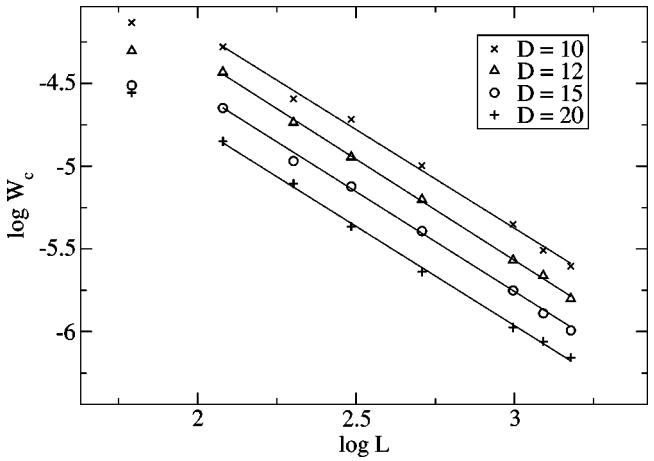

FIG. 2. Plot of natural logarithm of the critical damage fluctuations $W_c=(\langle p^2\rangle-\langle p\rangle^2)^{1/2}$ versus logarithm of system size L. The disorder is $D=10$, 12, 15, and 20, respectively, and the data for each different level of disorder are shifted vertically in order to separate them. The slopes are in that order for $D=10$: 1.19, $D=12$: 1.22, $D=15$: 1.20, and $D=20$: 1.20. Taking the uncertainty in the determination of each individual slope into account, we estimate $1/\nu=1.20\pm0.06$, giving $\nu=0.83\pm0.04$.

The currents are then calculated using the conjugate gradient algorithm [16]. After the currents i have been determined, the bond having the largest ratio $\max(i)/\max(t)$ is determined. This bond is then removed and the currents are recalculated. We do not allow the final crack to cross the plane along which the voltage drop is imposed. This simplifies the analysis of the final crack breaking the network apart, while it only imposes weak finite size corrections to fracture patterns.

The threshold values t constructed by setting $t=r^D$, where r is drawn from a uniform distribution on the unit interval [17]. This corresponds to a probability density $p(t)\propto t^{-1+\beta}$ on the interval $0<t<1$ with $\beta=1/D$. The parameter $D>0$ controls the width of the distribution: Larger values of D corresponds to stronger disorder. In order to ensure that our results are obtained in the strong disorder phase of the fuse model, we studied $D=10$, 12, 15, and 20. Our system sizes varied from $L=6-24$ with 20 000 samples generated for the smallest sizes to 2000 samples for the largest sizes.

With $D=20$, the smallest threshold values generated are of the order $(24^3)^{-20}\approx10^{-83}$. The system has, however, still not entered purely screened percolation regime. With this level of disorder, the system fails when a fraction of about 0.6 of the bonds have failed. The threshold values of the bonds that fail near the end of the process are about $0.6^{20}\approx10^{-4}$—which is of the order of the currents that are carried by the bonds in the system. Hence there is competition between threshold values and currents, making the failure process a correlated one rather than a pure percolation one even in this seemingly extreme case.

Figure 1 shows the damage profile in the current direction of the random fuse model with $D=10$. Each profile is the average over thousands of samples for small L to hundreds of samples for large L. We denote the (1,1,1) direction the z direction. We define the damage as the normalized average number of burned-out fuses in the plane orthogonal to the z direction at z. The distribution has a weak maximum in the middle. Such a maximum is much less pronounced for the stronger disorders (i.e., larger D values) we studied. It should be noted, however, that the averaging procedure we use emphasizes the region with the largest damage, making the visible maximum in Fig. 1 partly a result of the averaging process. We also point out that damage density for regions away from the maximum varies for different lattice sizes. This is most likely caused by statistical size effects in the averaging process when we apply it on finite lattices.

Following percolation analysis [7], we define the survival probability Π indicating the relative number of lattices that has survived for a given average damage p. Assuming that the disorder is broad enough so that p is independent of z and there is a finite critical value of $p=p_c$ at which 50% of the lattices survives, we have that

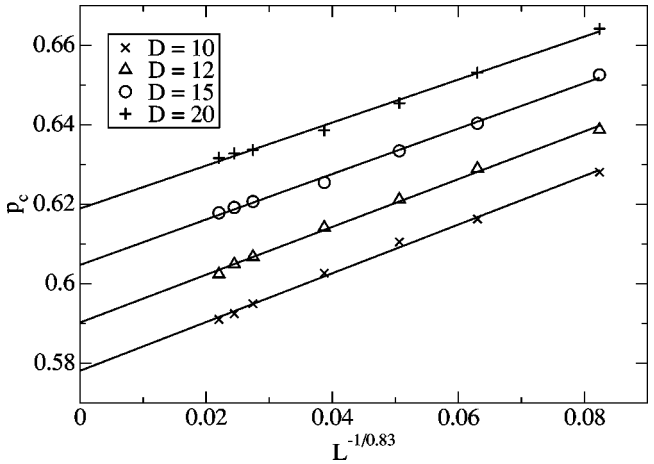

FIG. 3. $\langle p_c\rangle$ plotted against $L^{-1/\nu}$ with $\nu=0.83$. $D=10$ (×), $D=12$ (△), $D=15$ (○), and $D=20$ (+). As $L\to\infty$ the straight lines extrapolate to the thresholds $p_c=0.578$, $p_c=0.590$, $p_c=0.604$, and $p_c=0.619$, respectively.

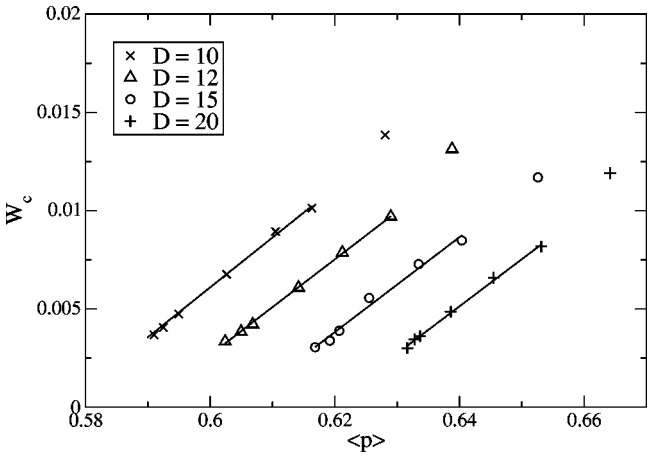

FIG. 4. We plot W_c as a function of $\langle p \rangle$. The data are the same as those used in Figs. 2 and 3.

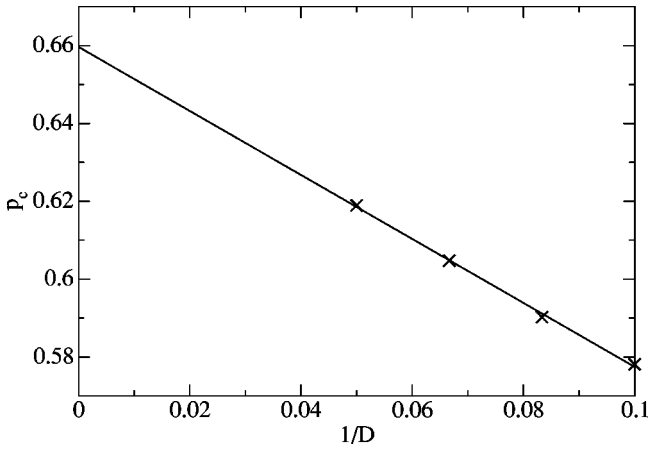

FIG. 5. p plotted against $1/D$ and extrapolated to infinite disorder giving $p_c(\infty)=0.66$. Extrapolating the straight line, towards increasing $1/D$ values, we find that p_c becomes negative for $1/D>0.75$.

$$\Pi = \Phi[(p - p_c)L^{1/\nu}]. \tag{5}$$

This scaling ansatz implies that both the mean value of the density of broken bonds $\langle p \rangle$ and the fluctuations $(\langle p^2 \rangle - \langle p \rangle^2)^{1/2}$ at breakdown scales as $L^{-1/\nu}$ using

$$\langle p \rangle = \int p \left(\frac{d\Pi}{dp} \right) dp, \tag{6}$$

and

$$\langle p^2 \rangle - \langle p \rangle^2 = \int (p - \langle p \rangle)^2 \left(\frac{d\Pi}{dp} \right) dp. \tag{7}$$

In Fig. 2 the fluctuations of the density of broken bonds, $W_c = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$, have been plotted against the system size L. The mean value of the slopes gives $\nu = 0.83 \pm 0.04$ which gives a roughness exponent $\zeta = 0.62 \pm 0.03$ in the three-dimensional fuse model when using Eq. (4). This is consistent with the previous measurements in Ref. [12].

We now turn to the scaling of $\langle p \rangle$. From finite-size scaling analysis, we expect the functional dependency

$$\langle p \rangle = p_c + \frac{A}{L^{1/\nu}} \tag{8}$$

on L where A is some positive constant. We show this relation for different values of D in Fig. 3.

Eliminating lattice size L between Eqs. (7) and (8) leads to a linear dependence between W_c and $\langle p \rangle$. We show this in Fig. 4.

This way of measuring the critical exponent ν is much less sensitive than the one presented in Fig. 2. From standard percolation in a simple cubic lattice the threshold for an infinite system is $p_c = 0.752$ [7]. The extrapolations done in Fig. 5 show results lying below this threshold. However, this is to be expected as the percolation process in this limit is *screened* [19]. This result strongly indicates that there is a strong disorder regime for finite disorders with p_c larger than zero in the three-dimensional fuse model. In fact, extrapolating the straight line in Fig. 5 towards larger $1/D$ values will result in p_c reaching zero and becoming negative at $D < 1.33$. This is physically impossible and p_c remains zero in this range. This indicates that there is a transition from a percolationlike regime with $p_c > 0$ for $D > 1.33$ to a regime with $p_c = 0$ for $D < 1.33$. This latter regime has been described as the diffuse localization regime in Refs. [11,5].

In summary, we have determined the correlation length exponent in the three-dimensional fuse model to be $\nu = 0.83 \pm 0.04$. Furthermore, using Eq. (4), this is consistent with the previously measured roughness exponent $\zeta = 0.62 \pm 0.05$ [12], lending support to the scenario proposed by Hansen and Schmittbuhl [4] for understanding the universality of the roughness exponent in the fuse model and brittle fracture. Our analysis was based on studying the fuse model with strong enough disorder for the breakdown process to develop in a percolationlike manner with p spationally stationary so that the tools developed for studying that problem could be used in the present one. We note that in this regime, one will not see the fracture roughness scaling of Eq. (1): The fracture will have a fractal structure. When, on the other hand, the disorder is weak enough for localization to set in, p is no longer spatially stationary, making a direct measurement of ν based on fluctuations in p impossible. However, it is in this regime fracture roughness scaling as in Eq. (1) is seen as shown in Ref. [12].

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